# ON $p-$ADIC INTEGERS AND THE ADDING MACHINE GROUP

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**Abstract**

In this paper, we define a natural metric on $Aut(X^*)$ and prove that the closure of the adding machine group, a subgroup of the automorphism group, is both isometric and isomorphic to the group of $p-$adic integers. So, we show that the group of $p-$adic integers can be isometrically embedded into the metric space $Aut(X^*)$.

## 1 Introduction

In recent years, there are many works on self-similar automorphism groups of the rooted tree $X^*$ ( [2], [4], [6]). The adding machine group is a typical example for self-similarity. We denote this group by $A$. $A$ is a cyclic group generated by

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where $a$ is an automorphism of the $p-$ary rooted tree and $\sigma = (012\ldots(p-1))$ is a permutation on $X = \{0, 1, 2, \ldots, (p-1)\}$. Thus, $A$ is isomorphic to $\mathbb{Z}$. On the other hand, one can consider the automorphism $a$ as adding one to a $p-$adic integer. That is why the term adding machine is used ( [4]). In [5], $p-$adic integers is pictured on a tree. This picture serves that any ultrametric space can be drawn on a tree.

In this paper, we equip $Aut(X^*)$ with a natural metric and prove that the group of $p-$adic integers is both isometric and isomorphic to the closure $\overline{A}$ of the adding machine group, a subgroup of the automorphism group of the $p-$ary rooted tree.

First we recall basic definitions and notions.

*$p-adic$ integers:* A $p-$adic integer is a formal series

$$\sum_{i\geq 0} a_i p^i$$

where each $a_i \in \{0, 1, 2, \ldots, (p-1)\}$ and the set of all $p-$adic integers is denoted by $\mathbb{Z}_p$.

Suppose that $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i\geq 0} b_i p^i$ be elements of $\mathbb{Z}_p$. Then $a$ addition with $b$, $c = \sum_{i\geq 0} c_i p^i$, is determined for each $m \in \{0, 1, 2, \ldots\}$ by

$$\sum_{i=0}^{m} c_i p^i \equiv \sum_{i=0}^{m} (a_i + b_i) p^i \quad (mod\ p^{m+1})$$

where $c_i \in \{0, 1, \ldots, (p-1)\}$. $\mathbb{Z}_p$ is a group under this operation and is called the group of $p-$adic integers.

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Let $a = \sum_{i \geq 0} a_i p^i$ be an element of $\mathbb{Z}_p$ and $a \neq 0$. Thus, there is a first index $v(a) \geq 0$ such that $a_v \neq 0$. This index is called the order of $a$ and is denoted by $ord_p(a)$. If $a_i = 0$ for $i = 0, 1, 2, \ldots$ then $ord_p(a) = \infty$. On the other hand, the $p-$adic value of $a$ is denoted by

$$|a|_p = \begin{cases} 0 & , \quad \text{if } a_i = 0 \text{ for } i = 0, 1, 2, \ldots, \\ p^{-ord_p(a)} & , \quad \text{otherwise} \end{cases}$$

and $d_p = |a - b|_p$ for $a, b \in \mathbb{Z}_p$ is a metric on $\mathbb{Z}_p$ (for details see [3], [7] and [8]).
*The automorphism group of the rooted tree:* Let $X$ be a finite set (alphabet) and let

$$X^* = \{x_1 x_2 \ldots x_n \mid x_i \in X, n \geqslant 0\ \}$$

be the set of all finite words. The length of a word $v = x_1 x_2 \ldots x_n \in X^*$ is the number of its letters and is denoted by $|v|$. The product of $v_1, v_2 \in X^*$ is naturally defined by concatenation $v_1 v_2$. One can think of $X^*$ as vertex set of a rooted tree.

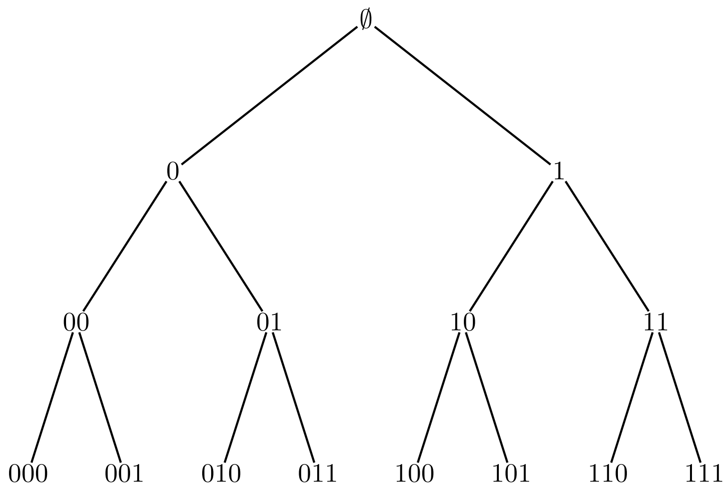


Figure 1.1: The first three levels of the binary rooted tree $X^*$ for $X = \{0, 1\}$

The set $X^n = \{v \in X^* \mid |v| = n\}$ is called the *nth* level of $X^*$. The empty word $\emptyset$ is the root of the tree $X^*$. Two words are connected by an edge if and only if they are of the form $v, vx$ where $v \in X^*$ and $x \in X$.

A map $f : X^* \to X^*$ is an endomorphism of the tree $X^*$ if it preserves the root and adjacency of the vertices. An automorphism is a bijective endomorphism. The group of all automorphisms of the tree $X^*$ is denoted by $Aut(X^*)$.

If $G \leq Aut(X^*)$ is an automorphism group of the rooted tree $X^*$ then for $v \in X^*$, the subgroup

$$G_v = \{g \in G \mid g(v) = v\}$$

is called the vertex stabilizer. The *nth* level stabilizer is the subgroup

$$St_G(n) = \bigcap_{v \in X^n} G_v.$$

We need a useful way to express automorphisms of the rooted tree $X^*$. For this aim, we give a definition and a proposition from [6].

**Definition 1.1** ( [6]). *Let $H$ be a group acting (from the right) by permutations on a set $X$ and let $G$ be an arbitrary group. Then the (permutational) wreath product $G \wr H$ is the semi-direct product $G^X \rtimes H$, where $H$ acts on the direct power $G^X$ by the respective permutations of the direct factors.*

Let $|X| = d$. The multiplication rule for the elements $(g_1, g_2, ..., g_d)h \in G \wr H$ is given by the formula

$$(g_1, g_2, ..., g_d)\alpha(h_1, h_2, ..., h_d)\beta = (g_1 h_{\alpha(1)}, g_2 h_{\alpha(2)}, ..., g_d h_{\alpha(d)})\alpha\beta$$

where $g_i, h_i \in G, \alpha, \beta \in H$ and $\alpha(i)$ is the image of $i$ under the action of $\alpha$.

**Proposition 1.2** ( [6]). *Denote by $S(X)$ the symmetric group of all permutations of $X$. Fix some indexing $\{x_1, x_2, ..., x_d\}$ of $X$. Then we have an isomorphism*

$$\psi : Aut(X^*) \to Aut(X^*) \wr S(X),$$

*given by*

$$\psi(g) = (g|_{x_1}, g|_{x_2}, ..., g|_{x_d})\alpha,$$

*where $\alpha$ is the permutation equal to the action of $g$ on $X \subset X^*$.*

Thus, $g \in Aut(X^*)$ is identified with the image $\psi(g) \in Aut(X^*) \wr S(X)$ and it is written as

$$g = (g|_{x_1}, g|_{x_2}, ..., g|_{x_d})\alpha.$$

*The adding machine group:* Let $a$ be the transformation on $X^*$ defined by the wreath recursion

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where $\sigma = (012\ldots(p-1))$ is an element of the symmetric group on $X = \{0, 1, 2, \ldots, (p-1)\}$. The transformation $a$ generates an infinite cyclic group on $X^*$. This group is called the adding machine group and we denote this group by $A$.

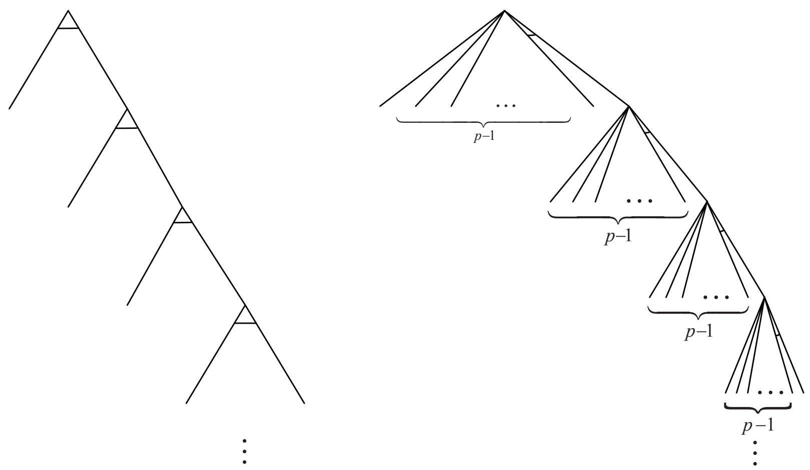


Figure 1.2: Portrait of the transformation $a$ for $X = \{0, 1\}$ and $X = \{0, 1, ..., p-1\}$

For example, using permutational wreath product we obtain that

$$\begin{aligned} a^p &= (1, \ldots, 1, a)\sigma(1, \ldots, 1, a)\sigma \ldots (1, \ldots, 1, a)\sigma \\ &= (a, a, \ldots, a)\sigma^p \\ &= (a, a, \ldots, a) \end{aligned}$$

(for details see [2], [6]).

# 2 The Metric Space ($Aut(X^*)$, $d$)

We define a natural metric on the automorphism group of the $p-$ary rooted tree $X^*$ where $X = \{0, 1, 2, ..., p-1\}$. This metric is used by [1].
**Definition 2.1.** *Let $g_1, g_2 \in Aut(X^*)$.*

$$d(g_1, g_2) = \begin{cases} \frac{1}{p^k} & \text{for } g_1^{-1}g_2 \in St_{Aut(X^*)}(k) \text{ and } g_1^{-1}g_2 \notin St_{Aut(X^*)}(k+1), \\ 0 & \text{for } g_1 = g_2. \end{cases}$$

*In other words, if $g_1$ and $g_2$ agree on all vertices of level $k$ but do not agree at least one vertex of level $(k+1)$ of the tree $X^*$ then the distance between $g_1$ and $g_2$ is $\frac{1}{p^k}$.*

($Aut(X^*)$, $d$) is a metric space. Moreover, it can easily be shown that the metric space ($Aut(X^*)$, $d$) is compact.

**Proposition 2.2.** *$Aut(X^*)$ is a topological group.*

*Proof.* First we prove that

$$\begin{array}{cccc} \psi & : & Aut(X^*) \times Aut(X^*) & \longrightarrow & Aut(X^*) \\ & & (g, h) & \longmapsto & gh \end{array}$$

is a continuous map. We take an arbitrary $(g_0, h_0) \in Aut(X^*) \times Aut(X^*)$. Let $U$ be a neighborhood of $g_0h_0$. There exists an integer $n$ such that

$$B\Big(g_0h_0, \frac{1}{p^n}\Big) = \Big\{f \mid d(f, g_0h_0) < \frac{1}{p^n}\Big\} \subseteq U.$$

We take an open set

$$V = V_1 \times V_2 = \{(g, h) \mid g \in V_1, h \in V_2\}$$

of $Aut(X^*) \times Aut(X^*)$ such that

$$V_1 = B\Big(g_0, \frac{1}{p^n}\Big) = \Big\{g \mid d(g, g_0) < \frac{1}{p^n}\Big\}$$

and

$$V_2 = B\Big(h_0, \frac{1}{p^n}\Big) = \Big\{h \mid d(h, h_0) < \frac{1}{p^n}\Big\}.$$

Now, we show that $\psi(V) \subseteq U$ where

$$\psi(V) = \psi(V_1 \times V_2) = \{gh \mid g \in V_1, h \in V_2\}.$$

Let $gh \in \psi(V)$. Thus, we have $g \in V_1$ and $h \in V_2$. Namely, we obtain that

$$g^{-1}g_0 \in St_{Aut(X^*)}(n+1) \text{ and } h^{-1}h_0 \in St_{Aut(X^*)}(n+1). \tag{1}$$

Furthermore, we get

$$(gh)^{-1}g_0h_0 = h^{-1}(g^{-1}g_0)h_0 \in St_{Aut(X^*)}(n+1).$$

From (1), $gh \in U$. Thus, $\psi$ is continuous. Similarly, we prove that

$$\begin{array}{cccc} \varphi & : & Aut(X^*) & \longrightarrow & Aut(X^*) \\ & & g & \longmapsto & g^{-1} \end{array}$$

is continuous. We take an arbitrary $g_0 \in Aut(X^*)$. Let $U$ be a neighborhood of $g_0^{-1}$. So, there exists an integer $n$ such that

$$B\Big(g_0^{-1}, \frac{1}{p^n}\Big) = \Big\{f \mid d(f, g_0^{-1}) < \frac{1}{p^n}\Big\} \subseteq U.$$

We take a neighborhood $V$ of $g_0$ in $Aut(X^*)$ such that

$$V = B\Big(g_0, \frac{1}{p^n}\Big) = \Big\{g \mid d(g, g_0) < \frac{1}{p^n}\Big\}.$$

Now, we show that $\varphi(V) \subseteq U$. Let $g^{-1} \in \varphi(V)$. Thus, we have $g \in V$. In other words,

$$gg_0^{-1} \in St_{Aut(X^*)}(n+1).$$

Due to the definition of $U$, $g^{-1} \in U$. That is, $\varphi$ is continuous. □

**Proposition 2.3.** *$\overline{A}$ is a subgroup of $Aut(X^*)$.*

*Proof.* We show that $gh \in \overline{A}$ and $g^{-1} \in \overline{A}$ for all $g, h \in \overline{A}$.

Suppose that $g, h \in \overline{A}$. This means that there are sequences $(g_n)$, $(h_n)$ in $A$ such that

$$\lim_{n\to\infty} g_n = g \text{ and } \lim_{n\to\infty} h_n = h.$$

Thus, it follows that $\lim_{n\to\infty}(g_n, h_n) = (g, h)$. On the other hand, we proved that

$$\begin{array}{ccccc} \psi & : & Aut(X^*) \times Aut(X^*) & \longrightarrow & Aut(X^*) \\ & & (g, h) & \longmapsto & gh \end{array}$$

is continuous. Hence, we have

$$\lim_{n\to\infty} g_n h_n = gh.$$

It follows that $gh \in \overline{A}$ since the sequence $g_n h_n \in A$. Similarly, because

$$\begin{array}{ccccc} \varphi & : & Aut(X^*) & \longrightarrow & Aut(X^*) \\ & & g & \longmapsto & g^{-1} \end{array}$$

is continuous we obtain

$$\lim_{n\to\infty} g_n^{-1} = g^{-1}.$$

That is, $g^{-1} \in \overline{A}$. Thus, $\overline{A}$ is a subgroup of $Aut(X^*)$. □

## 3 Embedding of the Group of $p-$adic Integers into the Automorphism Group of the $p-ary$ Rooted Tree

Now we give a formula for the distance between two elements of the adding machine group. Notice that this expression is similar to the distance between two elements of $p-$adic integers.

**Proposition 3.1.** *For $a^n, a^m \in A$, the distance $d(a^n, a^m)$ is*

$$\begin{array}{cccl} d & : & A \times A & \to \quad A \\ & & (a^n, a^m) & \mapsto \quad d(a^n, a^m) = \begin{cases} 0 & \text{for } n = m, \\ \frac{1}{p^k} & \text{for } n - m = tp^k, \end{cases} \end{array}$$

*where $t, k \in \mathbb{Z}$, $p$ is prime number and $(p, t) = 1$.*

*Proof.* First we compute $St_A(1)$. Using permutational wreath product we obtain that

$$\begin{aligned} a^p &= (1, 1, \dots, a)\sigma(1, 1, \dots, a)\sigma \dots (1, 1, \dots, a)\sigma \\ &= (a, a, \dots, a). \end{aligned}$$

Thus, $St_A(1) = \langle a^p \rangle$. Moreover, we get

$$\begin{aligned} a^{p^2} &= a^p a^p \dots a^p \\ &= (a, a, \dots, a)(a, a, \dots, a) \dots (a, a, \dots, a) \\ &= (a^p, a^p, \dots, a^p) \end{aligned}$$

We have $a^{p^2} \in St_A(2)$ because $a^p \in St_A(1)$. Therefore, $St_A(2) = \langle a^{p^2} \rangle$. By proceeding in a similar manner, we compute $St_A(k) = \langle a^{p^k} \rangle$.

So, elements of $A$ which are in $St_A(1)$ but are not in $St_A(2)$ can be expressed as

$$St_A(1) - St_A(2) = \{a^{tp} : (p,t) = 1\}$$

and in general, we have

$$St_A(k) - St_A(k+1) = \{a^{tp^k} : (p,t) = 1\}.$$

Let us take arbitrary $a^n, a^m \in A$. If $n = m$ then $a^n = a^m$ and $d(a^n, a^m) = 0$. Assume $n \neq m$. So there is a unique expression $n - m = tp^k$ such that $(p,t) = 1$. Then we obtain

$$a^{-m}a^n = a^{n-m} = a^{tp^k} \in St_A(k) - St_A(k+1)$$

and $d(a^n, a^m) = \frac{1}{p^k}$. □

**Proposition 3.2.** *Let $\sum_{i \geq 0} \alpha_i p^i \in \mathbb{Z}_p$. Then the sequence*

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$$

*is convergent.*

*Proof.* For any $\varepsilon > 0$, there is a positive integer $n_0$ such that $\frac{1}{p^{n_0}} < \varepsilon$. If $k > l$ and $k, l \geq n_0$ then it is obtained that

$$d(a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\alpha_0+\alpha_1 p+\ldots+\alpha_l p^l}) = \frac{1}{p^l} < \varepsilon.$$

from Proposition 3.1. Thus, it is a Cauchy sequence. Because $Aut(X^*)$ is a complete metric space, this sequence is convergent. □

Now we give our main proposition:

**Proposition 3.3.** *We define*

$$\varphi \ : \ \mathbb{Z}_p \ \rightarrow \ \overline{A}$$

*such that $\varphi(\sum_{i \geq 0} \alpha_i p^i)$ is the limit of the sequence $a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$. Then $\varphi$ is both an isometry and a group isomorphism.*

*Proof.* From Proposition 3.2, $\varphi$ is well-defined. Now we show that $\varphi$ is an isometry. In other words, we show that $d_p(\alpha, \beta) = d(\varphi(\alpha), \varphi(\beta))$ for every $\alpha, \beta \in \mathbb{Z}_p$. Let $\alpha = \sum_{i \geq 0} \alpha_i p^i$ and $\beta = \sum_{i \geq 0} \beta_i p^i$.

If $d_p(\alpha, \beta) = 0$ then we obtain $d(\varphi(\alpha), \varphi(\beta)) = 0$ since $\alpha_i = \beta_i$ for $i = 0, 1, 2, \ldots$.

If $d_p(\alpha, \beta) = \frac{1}{p^k}$ then $\alpha_i = \beta_i$ for $i < k$ and $\alpha_k \neq \beta_k$. We must show that $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Because $\varphi(\alpha)$ and $\varphi(\beta)$ are the limits of the sequences $a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$ and $a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots$ respectively, it is obtained that

$$\lim_{k \to \infty} (a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}) = (\varphi(\alpha), \varphi(\beta)).$$

Since any metric function is continuous,

$$d(a^{\alpha_0}, a^{\beta_0}), d(a^{\alpha_0+\alpha_1 p}, a^{\beta_0+\beta_1 p}), \ldots \rightarrow d(\varphi(\alpha), \varphi(\beta)).$$

From Proposition 3.1, we get

$$0, 0, ..., 0, \frac{1}{p^k}, \frac{1}{p^k}, \ldots, \frac{1}{p^k}, \ldots \rightarrow \frac{1}{p^k}.$$

So, we get $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Namely, $\varphi$ is an isometry map.

Moreover, $\varphi$ is injective since $\varphi$ is an isometry map.

Now we show that $\varphi$ is surjective. Let $b \in \overline{A}$ be arbitrary. Thus, there exists a sequence

$$a^{n_0}, a^{n_1}, \ldots, a^{n_k}, \ldots \to b$$

whose elements are in $A$. Furthermore, every integer $n_k$ can be expressed in $\mathbb{Z}_p$ as

$$\begin{array}{rcl} n_0 &=& \alpha_0^0 + \alpha_1^0 p + \alpha_2^0 p^2 + \ldots \\ n_1 &=& \alpha_0^1 + \alpha_1^1 p + \alpha_2^1 p^2 + \ldots \\ &\vdots& \\ n_k &=& \alpha_0^k + \alpha_1^k p + \alpha_2^k p^2 + \ldots \\ &\vdots& \end{array} \tag{2}$$

At least one of the numbers $0, 1, 2, ..., (p-1)$ occurs infinitely many times in the sequence $(\alpha_0^k)_k$. We choose one of them and denote it by $\beta_0$. Let $(\alpha_1^{k_l})_l$ be a subsequence of $(\alpha_1^k)_k$ such that $\alpha_0^{k_l} = \beta_0$ for $l = 0, 1, 2, \ldots$. Similarly, we denote by $\beta_1$, any one of the numbers that appears infinitely many times in the sequence $(\alpha_1^{k_l})_l$. Proceeding in this manner, we obtain a sequence

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, \ldots, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}, \ldots.$$

From Proposition 3.2, this sequence is convergent. Now we show this sequence converges to $b$. Due to the construction of (2), there exists a subsequence $(n_{k_s})$ of the sequence $(n_k)$ whose $p-$adic expression of term $s$th such that

$$\beta_0 + \beta_1 p + \beta_2 p^2 + \ldots + \beta_s p^s + \gamma_{s+1} p^{s+1} + \gamma_{s+2} p^{s+2} + \ldots$$

Hence, because

$$\lim_{s\to\infty} d(a^{\beta_0+\beta_1 p+\ldots+\beta_s p^s}, a^{n_{k_s}}) = 0$$

and from the triangle inequality, the sequence $(a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k})$ converges to $b$. So, $\varphi(\sum_{i\geq 0} \beta_i p^i) = b$ and $\varphi$ is surjective.

Finally, we prove that $\varphi$ is a homomorphism. In other words, we prove that

$$\varphi(\alpha+\beta) = \varphi(\alpha)\varphi(\beta)$$

for every $\alpha, \beta \in \mathbb{Z}_p$. Let $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots$, $\beta = \beta_0 + \beta_1 p + \beta_2 p^2 + \ldots$ and

$$\alpha + \beta = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \ldots.$$

From the definition of $\varphi$,

$$a^{\gamma_0}, a^{\gamma_0+\gamma_1 p}, a^{\gamma_0+\gamma_1 p+\gamma_2 p^2}, ... \to \varphi(\alpha+\beta).$$

Moreover, it follows that

$$a^{(\alpha_0+\beta_0)}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p+(\alpha_2+\beta_2)p^2}, \ldots \to \varphi(\alpha)\varphi(\beta)$$

since $Aut(X^*)$ is a topological group,

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots \to \varphi(\alpha)$$

and

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots \to \varphi(\beta).$$

In $\mathbb{Z}_p$,

$$\begin{array}{rcl} \alpha_0 + \beta_0 &=& \gamma_0 + \overline{\gamma_0} p + 0p^2 + 0p^3 + \ldots \\ \alpha_0 + \beta_0 + (\alpha_1+\beta_1)p &=& \gamma_0 + \gamma_1 p + \overline{\gamma_1} p^2 + 0p^3 + 0p^4 + \ldots \\ &\vdots& \\ \alpha_0 + \beta_0 + \ldots + (\alpha_k+\beta_k)p^k &=& \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} + 0p^{k+3} + \ldots. \\ &\vdots& \end{array}$$

Let $x = \alpha_0 + \beta_0 + \ldots + (\alpha_k + \beta_k)p^k$ and $y = \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} + 0p^{k+3} + \ldots$. Then we have

$$d(a^x, a^y) = \begin{cases} \frac{1}{p^k} & \text{if } \overline{\gamma_k} \neq 0, \\ 0 & \text{if } \overline{\gamma_k} = 0. \end{cases}$$

So we get $\varphi(\alpha + \beta) = \varphi(\alpha)\varphi(\beta)$ since

$$d(a^{\alpha_0+\beta_0}, a^{\gamma_0}), d(a^{\alpha_0+\beta_0+(\alpha_1+\beta_1)p}, a^{\gamma_0+\gamma_1 p}), \ldots \to d(\varphi(\alpha)\varphi(\beta), \varphi(\alpha+\beta))$$

and

$$\lim_{k\to\infty} d(a^x, a^y) = 0.$$

Thus the proof is completed. □

Consequently, the group of $p-$adic integers $\mathbb{Z}_p$ can be isometrically embedded into the metric space $Aut(X^*)$ since $\overline{A} \subseteq Aut(X^*)$.

**Example 3.4.** *We show $\varphi(-1)$ for $p = 2$ in Figure 3.1. It is well-known that*

$$-1 = 1 + 1.2^1 + 1.2^2 + ... + 1.2^k + ... \in \mathbb{Z}_2.$$

*Due to the definition of $\varphi$, $\varphi(-1)$ is the limit of the sequence*

$$a^1, a^{1+1.2^1}, a^{1+1.2^1+1.2^2}, ...$$

*in $A$ for $X = \{0, 1\}$. This limit equals to $a^{-1} = (a^{-1}, 1)\sigma$ because of Proposition 3.1.*

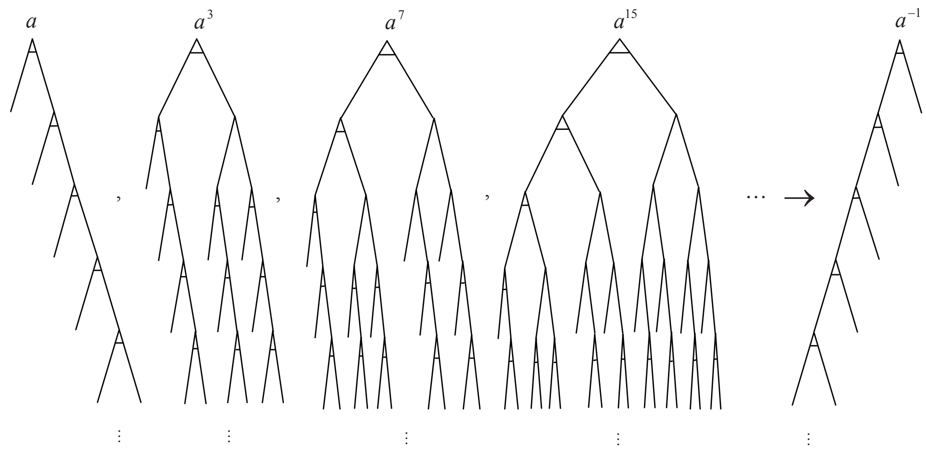


Figure 3.1: The image of $-1 \in \mathbb{Z}_2$ under the map $\varphi$

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